Answers to questions in the
Written Exam at the Department of Economics winter 2017-18

Economics of the Environment, Natural Resources and Climate Change

Final Exam

12 January, 2018

(3-hour closed book exam)

EXERCISE 1. Optimal natural resource extraction with pollution

Consider a model of the economy and the environment that uses the following notation:

$Y =$ production of final goods
$K =$ stock of man-made capital (physical and human)
$R =$ input of an exhaustible natural resource (raw material)
$S =$ reserve stock of the natural resource
$E =$ total emission of pollutant
$b =$ emission of pollutant per unit of raw material used in final goods production
$a =$ cost of extracting one unit of the natural resource (measured in units of the final good)
$C =$ consumption of final goods
$I =$ investment in produced capital
$U =$ lifetime utility of the representative consumer
$u =$ flow of utility per period
$\rho =$ rate of time preference
$t =$ time (treated as a continuous variable)

The lifetime utility of the representative consumer at time zero is

$$U_0 = \int_0^\infty u\left(C_t\right)e^{-\rho t}dt, \qquad u'>0, \quad u''<0, \quad \rho>0. \tag{1}$$

In the following, all variables except the constant parameters a, b and ρ will be understood to be functions of time, so for convenience we will generally skip the time subscripts.

The production of final goods is given by the production function

$$Y = F\left(K,R,E\right), \qquad F_K \equiv \frac{\partial F}{\partial K} > 0, \quad F_R \equiv \frac{\partial F}{\partial R} > 0, \quad F_E \equiv \frac{\partial F}{\partial E} < 0. \tag{2}$$

The assumption that $F_E < 0$ reflects that pollution has a negative impact on productivity.

Pollution is caused by the transformation of the raw material in the process of production. The use of one unit of raw material generates an emission of b units of the pollutant, so total emissions are

$$E = bR, \qquad b \text{ constant.} \tag{3}$$

The total cost of raw material production (measured in units of final goods) is aR, where a is the cost of extracting one unit of the natural resource. Hence the economy's aggregate resource constraint is

$$Y = C + I + aR, \qquad a \text{ constant.} \tag{4}$$

We will abstract from depreciation, so the net investment in man-made capital is equal to the gross investment I. Hence the change over time in the capital stock is

$$\dot{K} \equiv \frac{dK}{dt} = I. \tag{5}$$

We also abstract from the discovery of new reserves of the natural resource, so the change over time in the exhaustible resource stock is

$$\dot{S} \equiv \frac{dS}{dt} = -R. \tag{6}$$

The initial stocks of man-made and natural capital (K_0 and S_0) are predetermined.

Our first task is to characterize the first-best optimal allocation of resources that would be chosen by a benevolent social planner who maximizes the utility function (1) subject to the constraints implied by eqs. (2) through (6), taking K_0 and S_0 as given.

Question 1.1: Show that the current-value Hamiltonian corresponding to the social planner's problem may be written as

$$H = u(C) + \mu\left[F(K,R,bR) - C - aR\right] - \lambda R, \tag{7}$$

where μ is the shadow value of K, and λ is the shadow value of S. What are the control variables and what are the state variables in the social planner's optimal control problem?

Answer to Question 1.1: The general form of the current-value Hamiltonian is

$$H = u(C) + \mu \dot{K} + \lambda \dot{S}. \tag{i}$$

Inserting (2) and (5) in (4) and isolating $\dot{K}$ on the left-hand side, we get

$$\dot{K} = F(K,R,E) - C - aR. \tag{ii}$$

Inserting (ii) and (6) in (i) and using (3) to replace *E* by *bR*, we end up with (7). The control variables in the social planner's optimal control problem are *C* and *R*, and the state variables are *K* and *S*.

Question 1.2: Derive the first-order conditions for the solution to the social planner's optimal control problem.

Answer to Question 1.2: The first-order conditions are

$$\frac{\partial H}{\partial C} = 0 \quad \Rightarrow \quad u'(C) = \mu, \tag{iii}$$

$$\frac{\partial H}{\partial R}=0 \quad \Rightarrow \quad \mu\left(F_R+bF_E-a\right)=\lambda, \tag{iv}$$

$$\dot{\mu}=\rho\mu-\frac{\partial H}{\partial K} \quad \Rightarrow \quad \dot{\mu}=\mu\left(\rho-F_K\right), \tag{v}$$

$$\dot{\lambda}=\rho\lambda-\frac{\partial H}{\partial S} \quad \Rightarrow \quad \dot{\lambda}=\rho\lambda, \tag{vi}$$

$$\lim_{t\to\infty} e^{-\rho t}\mu_t K_t=0, \qquad \lim_{t\to\infty} e^{-\rho t}\lambda_t S_t=0. \tag{vii}$$

(Note: The transversality conditions in (vii) will not be used in the following, so it does not count as an error if they are not included in the answer to Question 1.1)

Question 1.3: Show that the first-order conditions for the solution to the social planner's problem imply the following condition for an optimal exploitation of the natural resource:

$$\dot{F}_R+b\dot{F}_E=\left(F_R+bF_E-a\right)F_K. \tag{8}$$

Give an economic interpretation of eq. (8) and explain the economic intuition behind it.

Answer to Question 1.3: Differentiating both sides of eq. (iv) with respect to time and remembering that *a* and *b* are constant, we get

$$\dot{\mu}\left(F_R+bF_E-a\right)+\mu\left(\dot{F}_R+b\dot{F}_E\right)=\dot{\lambda}. \tag{viii}$$

We can now insert (v) and (vi) in (viii) to eliminate $\dot{\mu}$ and $\dot{\lambda}$, respectively, thereby obtaining

$$\mu\left(\rho-F_K\right)\left(F_R+bF_E-a\right)+\mu\left(\dot{F}_R+b\dot{F}_E\right)=\rho\lambda. \tag{ix}$$

In the final step, we substitute (iv) in (ix) to eliminate λ to find

$$\mu\left(\rho-F_K\right)\left(F_R+bF_E-a\right)+\mu\left(\dot{F}_R+b\dot{F}_E\right)=\rho\mu\left(F_R+bF_E-a\right) \quad \Leftrightarrow$$

$$\left(\rho-F_K\right)\left(F_R+bF_E-a\right)+\dot{F}_R+b\dot{F}_E=\rho\left(F_R+bF_E-a\right). \tag{x}$$

The terms $\rho\left(F_R+bF_E-a\right)$ on the two sides of eq. (x) cancel out, leaving us with the equation

$$\dot{F}_R + b\dot{F}_E = \left(F_R + bF_E - a\right)F_K,$$

which is identical to eq. (8). This equation says that, in a social optimum, society should be indifferent between extracting an extra unit of the natural resource "today" or postponing extraction of an extra unit until "tomorrow". The left-hand side of (8) is the total gain from postponing extraction until "tomorrow" (i.e., for a very short while). The term $\dot{F}_R$ is the gain from the increase in the marginal productivity of the natural resource between today and tomorrow. The term $b\dot{F}_E$ is the gain from postponing extraction that arises in so far as the marginal productivity loss from the pollution caused by the use of raw materials will be smaller "tomorrow" than it is "today". Note that this gain could be either positive or negative, depending on the sign of $\dot{F}_E$. Thus the left-hand side of (8) is the total increase in the output of the final good made possible if the extraction and use of an extra unit of the raw material is postponed from today until tomorrow. The right-hand side of (8) measures the additional output of the final good that will be available tomorrow if an extra unit of raw material is extracted today and the resulting net increase in current output $F_R + bF_E - a$ is invested in capital. In that case the extra output available tomorrow will equal today's investment $F_R + bF_E - a$ multiplied by the rate of return on investment which is given by the marginal productivity of capital, F_K. It follows that, when condition (8) is met, society will be equally well off tomorrow whether it leaves an extra unit of the raw material underground today or whether it extracts it and invests the resource rent $F_R + bF_E - a$ in man-made capital. An equally valid interpretation of condition (8) is obtained if we divide by $F_R + bF_E - a$ on both sides of (8) to get

$$\frac{\dot{F}_R + b\dot{F}_E}{F_R + bF_E - a} = F_K. \tag{xi}$$

The left-hand side of (xi) is the marginal rate of return on investment in natural capital, where the investment takes the form of leaving an extra unit of the natural resource in the ground today in order to be able to extract and use an extra unit of raw material tomorrow. By postponing the extraction, society foregoes the possibility to consume the resource rent $F_R + bF_E - a$ today, so the denominator in the fraction on the left-hand side of (xi) is the amount invested in natural capital today. The absolute return on this investment is given by the numerator in (xi) which measures the increase in tomorrow's output obtained by postponing the use of an extra unit of raw material until tomorrow. The right-hand side of (xi) is the marginal rate of return on investment in man-made capital, given by its marginal productivity. Thus eq. (xi) says that the marginal rate of return on investment in natural capital should equal the marginal rate of return on investment in man-made capital to ensure an optimal composition of society's total stock of wealth which consists of the two forms of capital (*end of answer to Question 1.3*). (*Note: Only an exceptional student will be able to give an interpretation of condition (8) as elaborate as the one given above*).

We will now consider the resource allocation that will materialize in a market economy with private property and perfect competition in all markets. We start by focusing on the production of raw materials, and we assume that the reserves of natural resources are owned by private mining firms.

The market value of the representative mining firm at time zero is denoted by V_0^R, and the firm's payout of net dividends in the future period t is denoted by D_t^R. The market value of the firm is the present value of the future net dividends paid out to its owners, that is

$$V_0^R = \int_0^\infty D_t^R e^{-\int_0^t r_s ds} dt, \tag{9}$$

where r is the real market interest rate which may vary over time. The market price of the raw material (measured relative to the price of final goods) is p which also varies over time. The net dividend paid out by the mining firm in period t is therefore given by

$$D_t^R = \left(p_t - a\right) R_t. \tag{10}$$

The mining firm chooses its rate of raw material extraction R so as to maximize its market value (9) subject to (10), taking the price p as given, and accounting for the stock-flow relationship (6) between its current rate of extraction and its remaining reserve stock of the resource.

Question 1.4: Set up the current-value Hamiltonian for the mining firm's optimal control problem (you may denote the shadow price of the reserve stock by λ^R, and for convenience you may skip the time subscripts).

Answer to Question 1.4: Eqs. (6), (9) and (10) imply that the current-value Hamiltonian for the mining firm's optimal control problem (denoted by H^R) is

$$H^R = \left(p - a\right) R - \lambda^R R. \tag{xii}$$

Question 1.5: Derive the first-order conditions for the solution to the mining firm's problem.

Answer to Question 1.5: The mining firm's control variable is R and its state variable is S. Hence the first-order conditions are

$$\frac{\partial H^R}{\partial R} = 0 \quad \Rightarrow \quad p - a = \lambda^R, \tag{xiii}$$

$$\dot{\lambda}^R = r\lambda^R - \frac{\partial H^R}{\partial S} \quad \Rightarrow \quad \dot{\lambda}^R = r\lambda^R, \tag{xiv}$$

$$\lim_{t\to\infty} e^{-\int_0^t r_s ds} \lambda_t^R S_t = 0. \tag{xv}$$

(Note: It does not count as an error if the student has left out the transversality condition (xv) since this condition will not be used later on).

Question 1.6: Show that the mining firm's first-order conditions imply that

$$\dot{p} = \left(p - a\right)r. \tag{11}$$

Give an economic interpretation of eq. (11) and explain the economic intuition behind it.

Answer to Question 1.6: Differentiating both sides of (xiii) with respect to time and recalling that *a* is a constant, we get

$$\dot{p} = \dot{\lambda}^R. \tag{xvi}$$

Inserting (xiv) and (xiii) in (xvi), we get

$$\dot{p} = r\lambda^R = r\left(p - a\right),$$

which is identical to (11). Eq. (11) is a version of the Hotelling Rule for optimal exploitation of an exhaustible natural resource. The left-hand side is the extra revenue the resource owner can earn tomorrow as a result of a higher sales price if he postpones the extraction and sale of an extra unit of the resource from today until tomorrow. The right-hand side of (11) is the extra income accruing to the resource owner tomorrow if he extracts and sells an extra unit today and invests the resource rent $p - a$ in the capital market at the going interest rate *r*, thereby earning the additional capital income $\left(p - a\right)r$ tomorrow. When (11) is met the resource owner will therefore be indifferent between extracting an extra unit today or postponing the extraction until tomorrow, that is, his current rate of extraction will be privately optimal (*end of answer to Question 1.6*).

Consider next the representative firm producing the final good Y which we use as our numeraire good, thus setting its price equal to 1. The technology of the final goods firm is given by the production function (2), but since the representative firm is small relative to the market, it takes the aggregate flow of pollution E as given. The firm thus neglects its own contribution to total emissions and the resulting negative impact on the productivity of all firms. Its market value at time zero (V_0^Y) is the present value of its future net dividends,

$$V_0^Y = \int_0^\infty D_t^Y e^{-\int_0^t r_s ds} dt, \tag{12}$$

where D_t^Y is the net dividend paid out in the future period t. The government levies an environmental tax at the rate τ_t per unit of raw material used in final goods production, so the firm's total cost of a unit of raw material is $p_t + \tau_t$. Hence the net dividend paid out by the final goods firm is

$$D_t^Y = F\left(K_t, R_t, E_t\right) - \left(p_t + \tau_t\right)R_t - I_t. \tag{13}$$

The final goods firm chooses R and I with the purpose of maximizing its market value (12) subject to (13), taking the aggregate emission flow E as given, and accounting for the stock-flow relationship (5) between its investment and the change in its capital stock.

Question 1.7: Set up the current-value Hamiltonian for the optimal control problem of the final goods firm (you may denote the shadow price of its capital stock by μ^Y, and for convenience you may skip the time subscripts).

Answer to Question 1.7: From (12), (13) and (5) it follows that the current-value Hamiltonian for the optimal control problem of the final goods firm (denoted by H^Y) is

$$H^Y = F(K,R,E) - (p+\tau)R - I + \mu^Y I. \tag{xvii}$$

Question 1.8: Derive the first-order conditions for the solution to the problem of the final goods firm. Show that these conditions imply that

$$F_K = r, \tag{14}$$

$$\dot{p} = \dot{F}_R - \dot{\tau}. \tag{15}$$

Answer to Question 1.8: The firm's control variables are R and I, and its state variable is K. Hence the first-order conditions are

$$\frac{\partial H^Y}{\partial I} = 0 \quad \Rightarrow \quad \mu^Y = 1, \tag{xviii}$$

$$\frac{\partial H^Y}{\partial R} = 0 \quad \Rightarrow \quad F_R = p + \tau, \tag{xix}$$

$$\dot{\mu}^Y = r\mu^Y - \frac{\partial H^Y}{\partial K} \quad \Rightarrow \quad \dot{\mu}^Y = r\mu^Y - F_K, \tag{xx}$$

$$\lim_{t\to\infty} e^{-\int_0^t r_s ds} \mu_t^Y K_t = 0. \tag{xxi}$$

(Note: It does not count as an error if the student has left out the transversality condition (xxi) since this condition will not be used later on). From (xviii) it follows that

$$\dot{\mu}^Y = 0. \tag{xxii}$$

Inserting (xviii) and (xxii) into (xx), the result in (14) immediately follows. The result in (15) follows directly by differentiating both sides of (xix) with respect to time and rearranging the resulting expression.

Question 1.9: Use your findings in Question 1.6 and Question 1.8 to derive an expression for the optimal environmental tax rate τ which will ensure that resource extraction in the market economy will satisfy the condition (8) for a socially optimal rate of extraction. Explain the economic intuition for your result.

Answer to Question 1.9: In the answer to Question 1.6 we proved that (11) holds and in the answer to Question 1.8 we showed that (14) and (15) must be satisfied. Inserting (14) and (15) in (11) we get

$$\dot{F}_R - \dot{\tau} = (p - a) F_K. \tag{xxiii}$$

From (xix) we see that $p = F_R - \tau$ which may be substituted into (xxiii) to give

$$\dot{F}_R - \dot{\tau} = (F_R - \tau - a) F_K. \tag{xxiv}$$

Now suppose the government sets the tax rate on the use of raw materials in accordance with the formula

$$\tau = -bF_E. \tag{xxv}$$

We note that since $F_E < 0$, this tax rate is positive. Recalling that *b* is a constant, equation (xxv) implies that

$$\dot{\tau} = -b\,\dot{F}_E. \tag{xxvi}$$

When (xxv) and (xxvi) are inserted in (xxiv), we end up with the condition (8) for a socially optimal rate of natural resource extraction. The reason is that the Pigouvian tax rate specified in (xxv) fully internalizes the environmental externality caused by the use of raw materials in final goods production. The term $-bF_E$ is the marginal external cost of using an extra unit of raw material. This external cost is the output loss $-bF_E$ occurring when the processing of an extra unit of raw material increases the emission of the pollutant by the amount b, thereby reducing the productivity of all firms. When a tax rate equal to the marginal external cost of materials-related pollution is added to the market price of raw materials, final goods producers are confronted with the full marginal social cost of raw materials use. In the absence of other market distortions they will then use the optimal amount raw materials from a social viewpoint.

EXERCISE 2. Green tax reform

(Note: The questions in this exercise may be answered without any use of math and/or graphical analysis. However, you are welcome to use math or diagrams to the extent that you find it convenient).

In a “green” tax reform the government introduces (or raises) taxes on polluting goods and uses the revenue to lower taxes on income. In the following, you may assume that the green tax reform involves the introduction of a tax on one polluting good and that the revenue is used to lower a proportional tax on labour income.

Question 2.1: Explain the effects on economic efficiency (welfare) of a green tax reform (Hint: Is there a positive “second dividend”?)

Answer to Question 2.1: A revenue-neutral green tax reform that raises environmental taxes and reduces the labour income tax has the following effects: 1) *The environmental effect.* By reducing the consumption of polluting goods, the green tax reform improves the environment. The resulting welfare gain is often called “the first dividend” from a green tax reform. 2) *The revenue-recycling effect.* When the revenue from environmental taxes is used to finance a cut in the labour income tax, the distortionary effect of the latter tax is reduced. Ceteris paribus, this increases economic efficiency. 3) *The tax-interaction effect.* When the higher prices of polluting goods induce substitution away from consumption of these goods towards consumption of other goods, including consumption of leisure (implying a reduction in labour supply), the welfare cost of the pre-existing labour income tax is compounded.

The sum of the revenue-recycling effect and the tax-interaction effect is referred to as “the second dividend” from a green tax reform. The second dividend will typically be negative, i.e., the non-environmental welfare effect of a green tax reform will typically be negative. The intuition is that indirect taxes on polluting goods discourage labour supply by eroding the disposable real wage via higher consumer prices, just as the tax on labour income discourages labour supply by directly reducing the disposable real wage. Hence a shift from direct taxes on labour income to indirect taxes on polluting goods does not reduce the total tax burden on labour (unless the green taxes are allowed to reduce the real income of people outside the labour market so that more of the total tax burden is borne by these people such as pensioners, unemployed workers etc.). But in addition, the higher prices of polluting goods will induce substitution away from these goods towards other goods, thereby eroding the revenue from the pollution tax and reducing the scope for a cut in the labour income tax. The shift from the broad-based labour income tax to a narrow-based tax on a single polluting good will thus increase the scope for substitution away from taxed activities, making the tax system less efficient from a non-environmental perspective. More precisely, one can show that if the polluting good is a substitute for leisure, a revenue-neutral green tax reform will necessitate an increase in the total marginal effective tax rate on labour income (including indirect as well as direct taxes) because of the shift from the broad-based direct tax on labour income to the narrow-based indirect tax on the polluting good. This increase in the marginal effective tax rate on labour income will reduce labour supply, thus exacerbating the pre-existing tax distortion in the labour market.

Question 2.2: Will it be (second-best) optimal for a green tax reform to set the tax rate on the polluting good equal to the Pigouvian tax rate? Motivate your answer.

Answer to Question 2.2: In general it will be second-best optimal to set the tax rate on the polluting good below the Pigouvian level given by the marginal external cost generated by consuming an extra unit of the good. The reason is that, as explained in the answer to Question 2.1, a shift from the labour income tax to a tax on a polluting good generally compounds the pre-existing tax distortion in the labour market. To reduce this effect, it is optimal to sacrifice some of the environmental welfare gain from a Pigouvian tax in order to reduce the extra distortion to the labour market generated by a green tax reform.